Wireless Communication Systems

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I. Introduction

Communication system in a nutshell is about transmitting information from one place and receiving the information correctly in another place. If the medium which takes the transmitted signal to the receiver is free-space (air), then the system falls under wireless communication. In modern era, most communication takes form of digital communication. This means that we are exchanging information through bits. However, what would a stream of bits represent if a receiver and a transmitter does not share a common codebook? The received stream of bits in that case would be entirely useless.

II. Discrete Memoryless Channel

A. Continuous-Time Band-limited Gaussian Channel

Consider a system where input $x(t)$ and $z(t)$ are both continuous. The relationship between the two is described as

$$y(t) = x(t) + z(t), \tag{1}$$

where $x(t)$ is power constrained $\frac{1}{2T}\int_{-T}^{T} x^2(t)dt \leq P$, and $z(t)$ is continuous Gaussian white noise process, characterized by auto-correlation function,

$$r(\tau) = \mathbb{E}[z(t)z(t-\tau)] = \frac{N_o}{2}\delta(\tau). \tag{2}$$

This states that $z(t)$ and $z(t-\tau)$ are uncorrelated for $\tau \neq 0$. In frequency domain, $z(t)$ is characterized by it's power spectral density (PSD) described as

$$S(f) = \mathcal{F}\{r(\tau)\} = \frac{N_o}{2}, \tag{3}$$

which states that all the frequency components has constant $\frac{N_o}{2}$. How then, do we transmit a continuous signal through this continuous time channel? The idea is to look at the band-width limited version of the channel. We do this by applying $h(t)$, a low-pass filter, to $y(t)$ to produce $\tilde{y}(t)$. Since $\tilde{y}(t)$ is band-limited, we only need to consider band-limited $x(t)$; which allows us to use samples of $x(t)$ to represent $x(t)$. Sampling theorem states that if $x(t)$ is such that $X(f)$ is zero outside of $[-\omega, \omega]$, then sampled $x(t)$ at 2ω samples per second is sufficient for reconstructing $x(t)$ exactly.

Consider a system with continuous signal $x(t)$. We now want to sample this signal by multiplying it with impulse train described as

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{4}$$

Delta function is only non-zero $t = kT$ and zero else where ($t \neq kT$). In effect this creates a train of impulse (delta) functions at times kT, where k is integer from $-\infty$ to ∞. The sampling period $T = \frac{1}{2\omega}$, where ω is the highest frequency to represent $X(f)$. Therefore, $x_s(t) = x(t)p(t)$ effectively samples $x(t)$ at every sampling period T. Then, we can describe the sampled signal as

$$x_s(t) = x(t)p(t) \tag{5}$$

$$x_s(t) = x(kT)\sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{6}$$

Now, Fourier transform of $x_s(t)$ should give

$$X_s(f) = X(f) * P(f), \tag{7}$$

where $*$ is convolution. We know that

$$\mathcal{F}[p(t)] = P(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\big(f - \frac{k}{T}\big) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\big(f - 2\omega k\big). \tag{8}$$

This leads to

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(f - 2\omega k\big). \tag{9}$$

This means that $X(f)$ will repeat every 2ω in frequency domain. Note that it is crucial to sample at 2 times the maximum frequency component of $X(f)$ because the components will overlap if the sampling frequency is lower. Now, we have repeating

$X(f)$ every 2ω is frequency domain. We want to extract out one from each sample; therefore, we pass this with low-pass filter. This filter blocks out frequency components and only save the content in the interval $[-\omega, \omega]$. Description of this low pass filter is described as

$$Q(f) = \frac{1}{2\omega}, \text{ in } [\omega, \omega]. \tag{10}$$

To filter, you simply multiply $X_s(f)$ by (10) and recover

$$X(f) = X_s(f)Q(f). \tag{11}$$

Now, what is the time version of this filter? We can compute this by using Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df. \tag{12}$$

Now, we find the time version of $Q(f)$ to be,

$$q(t) = \int_{-\omega}^{\omega} \frac{1}{2\omega}e^{j2\pi ft}df = \frac{e^{j2\pi\omega t} - e^{-j2\pi\omega t}}{j2\pi t2\omega} = \frac{\sin(2\pi\omega t)}{2\pi\omega t} = \text{sinc}(t). \tag{13}$$

Putting it all together in time domain, you convolve (5) with (13) and get

$$\tilde{x}_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\text{sinc}(t - kT). \tag{14}$$

In effect, we have converted the continuous-time signal $x(t)$ to a discrete-time signal $x(kT)$.

Now, our input has become (14). For notation purpose, denote $x(kT)$ as x_n, which denotes n^{th} sample of $x(t)$. We pass this into the channel $z(t)$ and receive $y(t)$. However, how can we get back the samples of x_n from output now? We use matched filtering on $y(t)$ to get back x_n. We do this under the following knowledge that $\text{sinc}(t - kT)$ is a set of orthogonal basis functions; this is described by

$$\int_{-\infty}^{\infty} \text{sinc}\Big(t - \frac{n}{2\omega}\Big)\text{sinc}\Big(t - \frac{m}{2\omega}\Big)dt = \begin{cases} \frac{1}{2\omega} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \tag{15}$$

The match filtering then is described as

$$y_n = 2\omega \int_{-\infty}^{\infty} y(t)\text{sinc}\Big(t - \frac{n}{2\omega}\Big)dt \tag{16}$$

$$= 2\omega \int_{-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} x(kT)\text{sinc}(t - kT)\Big]\text{sinc}\big(t - nT\big)dt. \tag{17}$$

Hence, we get a discrete-time channel:

$$y_n = x_x + z_n, \tag{18}$$

where $z_n = 2\omega \int_{-\infty}^{\infty} z(t)\text{sinc}\big(t - \frac{n}{2\omega}\big)dt$.

B. Physical Wireless Channel Modeling

C. Complex Baseband Model

The wireless communication systems are modeled as linear time-invariant (LTI) system. For a system to be linear, it must satisfy the *superposition* and the *homogeneity* principle. Consider a linear system described as $T(.)$; the relationship between input x and output y is $y = T(x)$. Then, superposition is described as

$$y = T(x_1) + T(x_2) = T(x_1 + x_2), \tag{19}$$

which states that the response of a linear system to the sum of multiple inputs is equal to the sum of the responses of the system to each individual input. Further, homogeneity is explained by

$$\alpha y = T(\alpha x), \tag{20}$$

where same scalar α scales input as well as the output by the same amount.

Further, time invariant means that the system does not change over time. However, wireless communication channels change constantly over time because of many factors. Nonetheless, we can safely assume that the channel is time-invariant in channel coherence time; where coherence time is the time period where channel is approximately time-invariant. LTI system is described as

$$y_p(t) = (h_p * x_p)(t) = \int_{\infty}^{-\infty} h_p(u)x_p(t - u)du, \tag{21}$$

in time domain. Frequency representation of LTI system is

$$Y(f) = H(f)X(f). \tag{22}$$

III. Channel Capacity

We have a stream of bits that we want to transmit to the receiver. For this to be done, we package the whole bit stream into L separate symbols and send these in sequence. How would this particular communication system performance be measured? That is characterized by asking how many L symbols the packet contains, how many bits each of the symbol contains, and the probability of incorrect decoding at the receiver. These questions can be answered by channel capacity, which asks the question of what the maximum throughput (symbols per second or bits per symbol) of a communication system is.

Consider random variables X and Y. We put X through a channel and it produces Y. We are interested in knowing how much information from X can be carried over to Y through the channel; where the channel of the system is described by conditional distribution $f_{Y|X}(y|x)$. We know this question can be answered because of *channel coding theorem*; where it states that C (bit per symbol) is the capacity of the channel if for any given $\delta > 0$ and $\gamma > 0$, there exist a channel coding codebook of a finite length L that has rate $R = C - \gamma$ and offers an error probability $P(\text{error}) \leq \gamma$. We abstract away the practical method of finding this channel coding scheme, and move on with the belief that there exists such channel coding scheme. Then, we can formally find a way to derive channel capacity.

Channel capacity is formally defined as

$$C = \max_{f_X(x)} I(x;y), \tag{23}$$

where $I(x;y) = h(y) - h(y|x)$ is the mutual information between x and y. Hence, we want to maximize the mutual information over the distribution of X. The differential entropy of y is described as

$$h(y) = -\mathbb{E}[\log_2(f_Y(y))] \leq \log_2(\pi e Var(y)), \tag{24}$$

the equality holds if y is distributed as complex Gaussian random variable. Further, we define conditional differential entropy,

$$h(y|x) = -\mathbb{E}[\log_2(f_{Y|X}(y|x))]. \tag{25}$$

Now, consider that $x \sim \mathcal{CN}(0,p)$, where its probability density function is described as

$$f_X(x) = \frac{1}{\pi p} e^{-\frac{|x|^2}{p}}. \tag{26}$$

Utilizing this, we can compute the differential entropy of x,

$$\begin{aligned}
h(x) = -\mathbb{E}[\log_2(f_X(x))] &= -\int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} \log_2\left(\frac{1}{\pi p} e^{-\frac{|x|^2}{p}}\right) dx & (27)\\
&= \int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} \left(\log_2(\pi p) + \frac{|x|^2}{p}\log_2(e)\right) dx & (28)\\
&= \log_2(\pi p) \int_{\mathbb{C}} \frac{1}{\pi p} e^{-\frac{|x|^2}{p}} dx + \frac{\log_2(e)}{p} \int_{\mathbb{C}} \frac{|x|^2}{\pi p} e^{-\frac{|x|^2}{p}} dx & (29)\\
&= \log_2(\pi p) + \frac{\log_2(e)}{p}\mathbb{E}[|x|^2] = \log_2(\pi e p) & (30)
\end{aligned}$$

Now, we know assume that we know the distribution of x but what we do not know is distribution of y. However, the relationship between x and y is $y = gx + n$, where g is the wireless channel and $n \sim \mathcal{CN}(0, N_o)$ is noise. If we further assume that we know all the information about g, we can characterize the differential entropy of $(y|x)$ as

$$h(y|x) = [y - gx = n \sim \mathcal{CN}(0, N_o)] = \log_2(\pi e N_o). \tag{31}$$

Since addition of Gaussian random variable is also Gaussian, we know the distribution of $h(y)$ if we assume $x \sim \mathcal{CN}(0,p)$. Further, we know that this will give us the upper-bound of $h(y)$ because of (24). Therefore, we can differential entropy of y is

$$h(y) \leq [y = gx + n \sim \mathcal{CN}(0, q|g|^2 + N_o)] = \log_2(\pi e(q|g|^2 + N_o)). \tag{32}$$

We now have everything to calculate the channel capacity which is described as

$$C = h(y) - h(y|x) = \log_2\left(1 + \frac{q|g|^2}{N_o}\right). \tag{33}$$

IV. Modulation

We saw in sec. III, that we need input $x \sim \mathcal{CN}(0,p)$ to reach the capacity.

V. Estimation Theory

The objective of estimation theory is to estimate an unknown variable that cannot be directly observed, by utilizing an observation that has information about this unknown variable. There are two major methods in estimation theory: the classical and the Bayesian. In classical estimation, you consider the unknown variable to be a fixed variable where it does not change in time. In Bayesian estimation, you think of the unknown variable as a realization of a random variable, which means it can change. In wireless communication, estimation is usually required on estimating time-varying channels. Therefore, Bayesian approach is most likely the estimation method that you will choose.

VI. Wireless Channel

Is the wireless channel linear time-invariant? The linearity is given due to Maxwell's equations. However, time-invariance does not hold since channel changes over time. We still make analysis on channel as LTI because we consider the coherance time as time invariant. Coherence time is the time that channel is approximately time-invariant. We usually accept that coherence time $T_c = \frac{\lambda}{2v}$. It is important to note that coherence time is proportional to wavelength, meaning if frequency get higher, coherence time gets shorter.