Continuous-Time Band-limited Gaussian Channel

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I. CT Band-limited Channel

Consider a system where input $x(t)$ and $z(t)$ are both continuous. The relationship between the two is described as

$$y(t) = x(t) + z(t), \tag{1}$$

where $x(t)$ is power constrained $\frac{1}{2T}\int_{-T}^{T} x^2(t)dt \leq P$, and $z(t)$ is continuous Gaussian white noise process, characterized by auto-correlation function,

$$r(\tau) = \mathbb{E}[z(t)z(t-\tau)] = \frac{N_o}{2}\delta(\tau). \tag{2}$$

This states that $z(t)$ and $z(t-\tau)$ are uncorrelated for $\tau \neq 0$. In frequency domain, $z(t)$ is characterized by it's power spectral density (PSD) described as

$$S(f) = \mathcal{F}\{r(\tau)\} = \frac{N_o}{2}, \tag{3}$$

which states that all the frequency components has constant $\frac{N_o}{2}$. How then, do we transmit a continuous signal through this continuous time channel? The idea is to look at the band-width limited version of the channel. We do this by applying $h(t)$, a low-pass filter, to $y(t)$ to produce $\tilde{y}(t)$. Since $\tilde{y}(t)$ is band-limited, we only need to consider band-limited $x(t)$; which allows us to use samples of $x(t)$ to represent $x(t)$. Sampling theorem states that if $x(t)$ is such that $X(f)$ is zero outside of $[-\omega, \omega]$, then sampled $x(t)$ at 2ω samples per second is sufficient for reconstructing $x(t)$ exactly.

Consider a system with continuous signal $x(t)$. We now want to sample this signal by multiplying it with impulse train described as

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{4}$$

Delta function is only non-zero $t = kT$ and zero else where ($t \neq kT$). In effect this creates a train of impulse (delta) functions at times kT, where k is integer from $-\infty$ to ∞. The sampling period $T = \frac{1}{2\omega}$, where ω is the highest frequency to represent $X(f)$. Therefore, $x_s(t) = x(t)p(t)$ effectively samples $x(t)$ at every sampling period T. Then, we can describe the sampled signal as

$$x_s(t) = x(t)p(t) \tag{5}$$

$$x_s(t) = x(kT)\sum_{k=-\infty}^{\infty} \delta(t - kT). \tag{6}$$

Now, Fourier transform of $x_s(t)$ should give

$$X_s(f) = X(f) * P(f), \tag{7}$$

where $*$ is convolution. We know that

$$\mathcal{F}[p(t)] = P(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\big(f - \frac{k}{T}\big) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\big(f - 2\omega k\big). \tag{8}$$

This leads to

$$X_s(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X\big(f - 2\omega k\big). \tag{9}$$

This means that $X(f)$ will repeat every 2ω in frequency domain. Note that it is crucial to sample at 2 times the maximum frequency component of $X(f)$ because the components will overlap if the sampling frequency is lower. Now, we have repeating $X(f)$ every 2ω is frequency domain. We want to extract out one from each sample; therefore, we pass this with low-pass filter. This filter blocks out frequency components and only save the content in the interval $[-\omega, \omega]$. Description of this low pass filter is described as

$$Q(f) = \frac{1}{2\omega}, \text{ in } [\omega, \omega]. \tag{10}$$

To filter, you simply multiply $X_s(f)$ by (10) and recover

$$X(f) = X_s(f)Q(f). \tag{11}$$

Now, what is the time version of this filter? We can compute this by using Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df. \tag{12}$$

Now, we find the time version of $Q(f)$ to be,

$$q(t) = \int_{-\omega}^{\omega} \frac{1}{2\omega} e^{j2\pi f t} df = \frac{e^{j2\pi\omega t} - e^{-j2\pi\omega t}}{j2\pi t 2\omega} = \frac{\sin(2\pi\omega t)}{2\pi\omega t} = \mathrm{sinc}(t). \tag{13}$$

Putting it all together in time domain, you convolve (5) with (13) and get

$$\tilde{x}_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\mathrm{sinc}(t - kT). \tag{14}$$

In effect, we have converted the continuous-time signal $x(t)$ to a discrete-time signal $x(kT)$.

Now, our input has become (14). For notation purpose, denote $x(kT)$ as x_n, which denotes n^{th} sample of $x(t)$. We pass this into the channel $z(t)$ and receive $y(t)$. However, how can we get back the samples of x_n from output now? We use matched filtering on $y(t)$ to get back x_n. We do this under the following knowledge that $\mathrm{sinc}(t - kT)$ is a set of orthogonal basis functions; this is described by

$$\int_{-\infty}^{\infty} \mathrm{sinc}\Big(t - \frac{n}{2\omega}\Big)\mathrm{sinc}\Big(t - \frac{m}{2\omega}\Big) dt = \begin{cases} \frac{1}{2\omega} & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}. \tag{15}$$

The match filtering then is described as

$$y_n = 2\omega \int_{-\infty}^{\infty} y(t)\mathrm{sinc}\Big(t - \frac{n}{2\omega}\Big) dt \tag{16}$$

$$= 2\omega \int_{-\infty}^{\infty} \Big[\sum_{k=-\infty}^{\infty} x(kT)\mathrm{sinc}(t - kT)\Big]\mathrm{sinc}\big(t - nT\big) dt. \tag{17}$$

Hence, we get a discrete-time channel:

$$y_n = x_x + z_n, \tag{18}$$

where $z_n = 2\omega \int_{-\infty}^{\infty} z(t)\mathrm{sinc}\big(t - \frac{n}{2\omega}\big) dt$.